88087204

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Thursday 13 November 2008 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 12]*

(a) Show that the solution of the homogeneous differential equation

$$\frac{dy}{dx} = \frac{y}{x} + 1, \ x > 0,$$

given that $y = 0$ when $x = e$, is $y = x(\ln x - 1)$. *[5 marks]*

(b) (i) Determine the first three derivatives of the function $f(x) = x(\ln x - 1)$.

(ii) Hence find the first three non-zero terms of the Taylor series for $f(x)$ about $x = 1$. *[7 marks]*

2. *[Maximum mark: 19]*

(a) (i) Show that $\int_1^{\infty} \frac{1}{x(x+p)} dx$, $p \neq 0$ is convergent if $p > -1$ and find its value in terms of p.

(ii) Hence show that the following series is convergent.

$$\frac{1}{1 \times 0.5} + \frac{1}{2 \times 1.5} + \frac{1}{3 \times 2.5} + \cdots$$ *[8 marks]*

(b) Determine, for each of the following series, whether it is convergent or divergent.

(i) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n(n+3)}\right)$

(ii) $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{6}} + \sqrt{\frac{1}{12}} + \sqrt{\frac{1}{20}} + \cdots$ *[11 marks]*

3. *[Maximum mark: 12]*

The function $f(x) = \frac{1+ax}{1+bx}$ can be expanded as a power series in x, within its radius of convergence R, in the form $f(x) \equiv 1 + \sum_{n=1}^{\infty} c_n x^n$.

(a) (i) Show that $c_n = (-b)^{n-1}(a-b)$.

(ii) State the value of R. *[5 marks]*

(b) Determine the values of a and b for which the expansion of $f(x)$ agrees with that of e^x up to and including the term in x^2. *[4 marks]*

(c) Hence find a rational approximation to $\mathrm{e}^{\frac{1}{3}}$. *[3 marks]*

4. *[Maximum mark: 17]*

(a) Show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x \cos^2 y,$$

given that $y = \frac{\pi}{4}$ when $x = \pi$, is $y = \arctan(1 + \sin x)$. *[5 marks]*

(b) Determine the value of the constant a for which the following limit exists

$$\lim_{x \to \frac{\pi}{2}} \frac{\arctan(1+\sin x) - a}{\left(x - \frac{\pi}{2}\right)^2}$$

and evaluate that limit. *[12 marks]*